Lecture 2: Complex Numbers cont. & 1.2 Polar Form

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Recall:

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

Remember:

(i) $i^2 = -1$

(ii) To divide $a + bi$, expand the fraction with $a = bi$

The Conjugate

notation for the conjugate

Let $z = a + bi$. Then, $\bar{z} := a - bi$ → "complete conjugate of z"

$|z| := \sqrt{a^2 + b^2}$ → "modulus of z" or "absolute value of z"

Observe:

$z * \bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$

Useful properties Let $z, w \in \mathbb{C}$, then:

(i) $\overline{z + w} = \bar{z} + \bar{w}$

(ii) $\overline{zw} = \bar{z} * \bar{w}$

(iii) $\bar{\bar{z}} = z$

(iv) $\bar{z} = z \iff \mathrm{Im}(z) = 0 \iff z \in \mathbb{R}$

if and only if

(v) $\bar{z} = -z \iff \mathrm{Re}(z) = 0 \iff$: z is purely imaginary

(vi) $|z| \in \mathbb{R}$ with $|z| \geq 0$

Moreover, $|z| = 0 \iff z = 0$

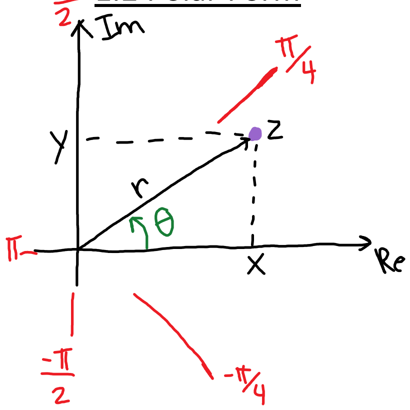
☆ <=> represents if and only if

(vii) $|z| = |\bar{z}|$

(viii) $|zw| = |z| * |w|$

(ix) $\underbrace{|z + w| \leq |z| + |w|}_{\text{"triangle inequality"}}$

1.2 Polar Form



$z = a + bi$

Describe z in terms of:

(1) $|z| \geq 0$

(2) argument $Arg(z) = \theta$

with $-\pi \leq \theta \leq \pi$

look from O into positive real axis and measure the angle counterclockwise

"argument"

So if $z = x + yi \in \mathbb{C}$ then

$$\frac{x}{r} = \cos(\theta), \quad \frac{y}{r} = \sin(\theta)$$

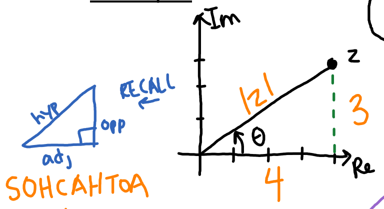
and

$$r = |z| = \sqrt{x^2 + y^2}$$

Therefore:

$$z = x + yi = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)).$$

<u>Example</u> $z = 4 + 3i$



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$$sin\theta = \frac{y}{r} = \frac{Im(z)}{|z|}$$

$$cos\theta = \frac{x}{r} = \frac{Re(z)}{|z|}$$

$$tan\theta = \frac{y}{x} = \frac{Im(z)}{Re(z)}$$

(1) $|z| = \sqrt{16 + 9}$
$= 5$

θ=?

$$sin\theta = \frac{3}{5}$$

$$cos\theta = \frac{4}{5}$$ (2)

$\theta \approx 36.87°$

$$z = Re(z) + i * Im(z)$$
$$= |z| * (cos\theta + i * sin\theta)$$

Polar form (version 1)

However, Euler proved that by comparing power series with trig functions, one will discover:
$e^{i\theta} = \cos(\theta) + i * sin(\theta)$, which means that we can modify our original polar form to:
$z = |z| * (\cos(\theta) + i * \sin(\theta))$
$= |z| * e^{i\theta}$ (Polar form version 2)

<u>Observations</u>:

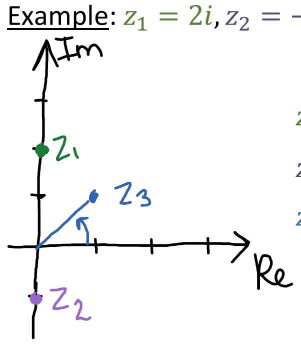
(1) If $r > 0$, then $r * e^{i\theta} = r' * e^{i\theta}$
|z| $\quad$ <=>$r = r'$ and $\theta = \theta' + 2\pi n \;\; when \;\; n\epsilon\mathbb{Z}$

(2) $\overline{r + e^{i\theta}} = r * e^{i\theta}$

(3) $|e^{i\theta}| = 1$

$Polar\ form\text{:}\ |z| * e^{i\theta}$

<u>Example</u>: $z_1 = 2i, z_2 = -i, z_3 = 1 + i$



$z_1 = 2 * e^{i*\frac{\pi}{2}}$ → 90° = π/2

$z_2 = 1 * e^{i*\frac{3\pi}{2}}$

$z_3 = \sqrt{2} * e^{i*\frac{\pi}{4}}$

<u>Multiplication in Polar Form</u>

$z = r * e^{i\theta} \qquad w = t * e^{i\alpha}$

$$z * w = r(\cos\theta + i * sin\theta) * t(cos\alpha + i * sin\alpha)$$
$$= rt[(cos\theta cos\alpha - sin\theta sin\alpha) - i(cos\theta sin\alpha + sin\theta cos\alpha)]$$
$$= rt[\cos(\theta + \alpha) + isin(\theta + \alpha)]$$
$$= rt * e^{i(\theta+\alpha)}$$

$= rt\lfloor(cos\theta cos\alpha - sin\theta sin\alpha) - i(cos\theta sin\alpha + sin\theta cos\alpha)\rfloor$
$= rt[\cos(\theta + \alpha) + isin(\theta + \alpha)]$
$= rt * e^{i(\theta+\alpha)}$

Multiply the moduli and add the arguments

Just like usual multiplication with exponents!